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Online Publication Date: 01 June 1996

To cite this Article: Hollerbach, Rainer (1996) 'Magnetohydrodynamic shear layers in a rapidly rotating plane layer', Geophysical & Astrophysical Fluid Dynamics, 82:3, 237 - 253

To link to this article: DOI: 10.1080/03091929608213637

URL: http://dx.doi.org/10.1080/03091929608213637

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MAGNETOHYDRODYNAMIC SHEAR LAYERS IN A RAPIDLY ROTATING PLANE LAYER

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(Received 23 August 1995; in final form 15 December 1995)

We consider free shear layers in a rapidly rotating fluid layer defined by \( |z| < 1 \). At \( z = \pm 1 \) we impose the velocity \( v = F_1(x) \mp F_3(x) \). Discontinuities in \( F_1 \) and \( F_3 \) then turn out to induce vertical shear layers throughout the fluid. A discontinuity in \( F_1 \) induces two layers of thicknesses \( E^{1/4} \) and \( E^{1/3} \), a discontinuity in \( F_3 \) induces one layer of thickness \( E^{1/3} \). The Ekman number \( E \) is an inverse measure of the rotation rate. We demonstrate that these shear layers are in fact the same as the classical Stewartson layers in cylindrical geometry. We next consider the effect of imposing a magnetic field across these shear layers. In the limit of small magnetic Reynolds number \( R_m \), we demonstrate that for \( \Lambda \gtrsim E^{1/3} \) all of the previous shear layer structure is ultimately suppressed, where the modified Elsasser number \( \Lambda \equiv \Lambda R_m \) is a measure of the strength of the imposed field. The various transitions between the non-magnetic and the fully magnetic regimes are discussed in some detail.

KEY WORDS: Earth's core, magnetohydrodynamics, shear layers.

1. INTRODUCTION

Proudman (1956) and Stewartson (1957, 1966) have considered the flow of an incompressible fluid confined between two concentric spheres rotating about the same axis with almost the same angular velocity. In the asymptotic limit of rapid rotation, they obtained a rather intricately nested shear layer on the cylinder parallel to the rotation axis and tangent to the inner sphere. This shear layer has been reproduced numerically by Hollerbach (1994), who also investigated the effect that an imposed magnetic field would have on it in an electrically conducting fluid. It is of particular interest to consider this magnetohydrodynamic shear layer, in that the Earth's magnetic field is created in a rotating spherical shell of electrically conducting fluid, its liquid iron outer core. The geophysical aspects of this shear layer have been considered by Ruzmaikin (1989).

In this work we consider these magnetohydrodynamic shear layers analytically, rather than numerically, thereby placing the qualitative results of Hollerbach (1994) on a somewhat firmer basis. It is unfortunately beyond my abilities to solve this problem in the proper spherical geometry, although this has been attempted by Kleeorin et al. (1995). We will solve it in a simpler geometry instead. In a similar spirit, Stewartson (1957) first solved the non-magnetic problem in cylindrical geometry. Even that turns out to be too difficult for the magnetic problem, for reasons given below, and so we will solve it in Cartesian geometry. We will begin by obtaining the non-magnetic shear
layers, and demonstrating that they are exactly the same as Stewartson's (1957) layers in cylindrical geometry, which in turn are almost the same as Stewartson's (1966) layers in spherical geometry. We will then consider the effect of imposing a magnetic field across these shear layers, and demonstrate that a sufficiently strong field will suppress them, as already noted by Hollerbach (1994). Here, however, we will be able to give a precise asymptotic answer as to what constitutes a "sufficiently strong" field.

2. NON-MAGNETIC SHEAR LAYERS

Basic Equations

We begin by considering the non-magnetic shear layers. The main purpose of this section is to introduce the basic model and the method of analysis, and to demonstrate that the shear layers one obtains in this Cartesian geometry are identical with the shear layers Stewartson (1957) obtained in cylindrical geometry.

Consider the geometry depicted in Figure 1: Fluid is confined in a rapidly rotating plane layer defined by $|z| < 1$. In the rotating reference frame, the suitably non-dimensionalized momentum equation is then simply

$$2k \times \mathbf{U} = -\nabla p + EV^2 \mathbf{U},$$

(2.1)

where the (small) Ekman number $E$ is an inverse measure of the rotation rate. To obtain a shear layer, indeed, to obtain any motion at all, we impose on the planes $z = \pm 1$ the motion $v = F_s(x) \pm F_a(x)$, as indicated in Figure 1. (These forcing functions $F_s$ and $F_a$ are thus symmetric and antisymmetric, respectively, with respect to the midplane $z = 0$, but without necessarily having any particular symmetry in $x$.) The Cartesian coordinates $(x, y, z)$ thus correspond to Stewartson's cylindrical coordinates $(r, \theta, z)$.

Figure 1  A sketch of the basic geometry. The fluid is confined within the plane layer defined by $|z| < 1$. At $z = \pm 1$ the motion $v = F_s(x) \pm F_a(x)$ is imposed. The uniform magnetic field $\mathbf{B}_0$ that will be imposed in the next section is also shown.
By symmetry, the entire motion will then be independent of $y$, and so we can use the incompressibility condition $\nabla \cdot U = 0$ to decompose $U$ as

$$U = \nabla \times (\psi \hat{j}) + v \hat{j}. \quad (2.2)$$

The momentum equation (2.1) then becomes

$$\frac{\partial}{\partial z} v + E \nabla^2 v = 0, \quad (2.3a)$$

$$\frac{\partial}{\partial z} v - E \nabla^4 \psi = 0, \quad (2.3b)$$

and the associated boundary conditions at $z = \pm 1$ become

$$v = F_s(x) \pm F_a(x), \quad (2.4a)$$

$$\psi = \frac{\partial}{\partial z} \psi = 0. \quad (2.4b)$$

For convenience, we will also take $F_s$ and $F_a$ to be of period $2\pi$, so that we may restrict attention to the interval $(-\pi, \pi)$. In particular, we will wish to consider $F_s$ and $F_a$ of the form

$$F = \begin{cases} x + \pi, & -\pi < x < 0, \\ x - \pi, & 0 < x < \pi, \end{cases} \quad (2.5)$$

that is, a sawtooth wave with a discontinuity at $x = 0$. These discontinuities will then turn out to induce shear layers at $x = 0$. The assumption of periodicity is not essential; it merely allows us to work with discrete Fourier sums rather than continuous Fourier integrals.

**Exact Solution**

Adapting Stewartson's (1957) expansion to Cartesian geometry, we expand $v$ and $\psi$ as

$$v = \sum_{k = -\infty}^{\infty} \left[ A^s(k) e^{ikx} \cosh(xz) + A^a(k) e^{ikx} \sinh(xz) \right], \quad (2.6a)$$

$$\psi = \sum_{k = -\infty}^{\infty} \left[ B^s(k) e^{ikx} \sinh(xz) + B^a(k) e^{ikx} \cosh(xz) \right]. \quad (2.6b)$$

These expansions will satisfy (2.3) if

$$2\pi B + E(\pi^2 - k^2) A = 0, \quad (2.7a)$$

$$2\pi A - E(\pi^2 - k^2)^2 B = 0, \quad (2.7b)$$
for both the symmetric and antisymmetric coefficients $A$ and $B$, separately. In order for (2.7) to have non-trivial solutions, $x$ must satisfy exactly the same equation,

$$E^2(x^2 - k^2)^3 = -4x^2,$$

(2.8)

as in Stewartson’s cylindrical geometry. (Note, incidentally, that Stewartson chose to work in terms of the Reynolds number $R = E^{-1}$.) This cubic in $x^2$ then has three roots $x_1, x_2, x_3$, with positive real parts.

The general solution to (2.3) is thus

$$v = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{3} \left[ A_n^s(k) e^{ikx} \cosh(x_nz) + A_n^a(k) e^{ikx} \sinh(x_nz) \right],$$

(2.9a)

$$\psi = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{3} \left[ B_n^s(k) e^{ikx} \sinh(x_nz) + B_n^a(k) e^{ikx} \cosh(x_nz) \right],$$

(2.9b)

where

$$B_n = -E \frac{x_n^2 - k^2}{2\alpha_n} A_n.$$ 

(2.9c)

For each Fourier mode $k$, the boundary conditions (2.4) then yield

$$\sum_{n=1}^{3} \cosh(x_n) A_n^s(k) = f_s(k),$$

(2.10a)

$$\sum_{n=1}^{3} \frac{x_n^2 - k^2}{\alpha_n} \sinh(x_n) A_n^a(k) = 0,$$

(2.10b)

$$\sum_{n=1}^{3} (\alpha_n^2 - k^2) \cosh(x_n) A_n^s(k) = 0,$$

(2.10c)

and

$$\sum_{n=1}^{3} \sinh(x_n) A_n^a(k) = f_a(k),$$

(2.11a)

$$\sum_{n=1}^{3} \frac{x_n^2 - k^2}{\alpha_n} \cosh(x_n) A_n^a(k) = 0,$$

(2.11b)

$$\sum_{n=1}^{3} (x_n^2 - k^2) \sinh(x_n) A_n^a(k) = 0,$$

(2.11c)

where $f_s(k)$ and $f_a(k)$ are the Fourier coefficients of $F_s(x)$ and $F_a(x),$

$$F(x) = \sum_{k=-\infty}^{\infty} f(k) e^{ikx}.$$ 

(2.12)
For each \( k \) then, the three equations (2.10) determine the three symmetric coefficients \( A_s^i(k) \), and the three equations (2.11) determine the three antisymmetric coefficients \( A_a^i(k) \), yielding an exact formal solution to the problem.

**Asymptotic Analysis**

Again following Stewartson, we wish to consider the asymptotic behaviour of this exact solution in the limit of small \( E \). Since the characteristic equation (2.8) for \( \alpha^2 \) is the same as Stewartson’s, so are its roots

\[
\alpha_1 \approx E k^{3/2}, \quad \alpha_2 \approx E^{-1/2}(1 + i), \quad \alpha_3 \approx E^{-1/2}(1 - i), \quad (2.13a,b,c)
\]

valid provided that \( k \ll E^{-1/2} \). Note then that \( \alpha_1 \ll k \), whereas \( \alpha_2, \alpha_3 \gg k \). Thus, \( \alpha_1 \) will turn out to be associated with regions of rapid variation in \( x \), namely the vertical shear layers, whereas \( \alpha_2 \) and \( \alpha_3 \) will turn out to be associated with regions of rapid variation in \( z \), namely the top and bottom Ekman layers.

Having obtained these approximate roots (2.13), the boundary conditions (2.10) and (2.11) then yield the approximate coefficients

\[
A_s^i(k) \approx k f_s(k)/[2E^{-1/2}\sinh(\alpha_1) + kcosh(\alpha_1)], \quad (2.14a)
\]

\[
A_a^i(k) \approx k f_a(k)/[2E^{-1/2}\cosh(\alpha_1) + ksinh(\alpha_1)]. \quad (2.14b)
\]

Again, the coefficients \( A_2 \) and \( A_3 \) are associated exclusively with the Ekman layers, and will therefore not be considered in the same detail as the shear layer coefficients \( A_1 \).

If \( f_s(x) \) is sufficiently smooth that \( f_s(k) \) is significant only for \( k \ll E^{-1/4} \), then (2.14a) simplifies further to

\[
A_s^i(k) \approx f_s(k), \quad (2.15)
\]

and so, again away from the Ekman layers where \( A_s^i \) and \( A^a_i \) also contribute, the symmetric component of the flow is

\[
v \approx \sum_k f_s e^{ikx}\cosh(\alpha_1 z) \approx F_s(x), \quad (2.16a)
\]

\[
\psi \approx \sum_k f_a e^{ikx}\sinh(\alpha_1 z) \approx E^{1/2} \sum_k k^2 f_a e^{ikx}. \quad (2.16b)
\]

Similarly, if \( f_a(x) \) is sufficiently smooth that \( f_a(k) \) is significant only for \( k \ll E^{-1/3} \), then (2.14b) simplifies further to

\[
A_a^i(k) \approx E^{1/2} k f_a(k)/2, \quad (2.17)
\]
and so, again away from the Ekman layers where \( A^{a}_z \) and \( A^{e}_z \) also contribute, the antisymmetric component of the flow is

\[
v \approx \sum_k E^{1/2} \frac{k f_a}{2} e^{ikx} \sinh(z_1 z) \approx E^{3/2} \frac{z}{4} \sum_k k^4 f_a e^{ikx}, \tag{2.18a}
\]

\[
\psi \approx \sum_k E^{1/2} \frac{f_a}{2} e^{ikx} \cosh(z_1 z) \approx \frac{E^{1/2}}{2} F_a(x). \tag{2.18b}
\]

Of course, what we are really interested in is the structure when \( F_a(x) \) and \( F_e(x) \) are not so smooth, as in (2.5), for which \( f(k) = i/k \) is never exponentially small. According to (2.14), \( A^e \) and \( A^a \) will nevertheless be exponentially small for all \( k \geq E^{-1/3} \), for which \( z_1 \gtrsim O(1) \). This is the origin of the \( E^{1/3} \) shear layers, just as in Stewartson (1957). Even if \( F_a(x) \) or \( F_e(x) \) exhibit structure on scales finer than \( E^{1/3} \), this will be suppressed, and only structure as fine as \( E^{1/3} \) will appear in the shear layer solution.

For the antisymmetric forcing \( F_a(x) \), this \( E^{1/3} \) shear layer is all there is, since (2.17) is valid right up to \( k \lesssim E^{-1/3} \). However, for the symmetric forcing \( F_e(x) \), there is an additional \( E^{1/4} \) shear layer, since (2.15) is valid only up to \( k \leq E^{-1/4} \). For \( k \gtrsim E^{-1/4} \), \( 2E^{-1/2} \sinh(z_1) \) begins to dominate \( k \cosh(z_1) \) in the denominator of (2.14a), in contrast with the denominator of (2.14b), where \( 2E^{-1/2} \cosh(z_1) \) always dominates \( k \sinh(z_1) \). This shift in the denominator of \( A^e \) is then the origin of the \( E^{1/4} \) shear layer, again just as in Stewartson (1957).

In other words, in the denominator of (2.14b), there are only two possible scalings: for \( k \lesssim E^{-1/3} \) it is essentially constant, and for \( k \gtrsim E^{-1/3} \) it is exponentially large. Because there is only one shift in the scaling, there is only one shear layer. In contrast, in the denominator of (2.14a), there are three possible scalings: for \( k \lesssim E^{-1/4} \) it is linear in \( k \), for \( E^{-1/4} \lesssim k \lesssim E^{-1/3} \) it is cubic in \( k \), and for \( k \gtrsim E^{-1/3} \) it is again exponentially large. Because there are now two shifts in the scaling, there are also two shear layers. In the next section we will demonstrate how the presence of a magnetic field affects these shifts in the scalings, and thus the structure of the shear layers.

**Examples**

But first, on the theory that a picture is worth a thousand words, we will present two examples illustrating the structure of these shear layers. We will present horizontal slices through the shear layers, well away from the Ekman layers. Away from the Ekman layers, the vertical structure does indeed turn out to be essentially as in (2.16) and (2.18), even within the shear layers.

We take, in turn, \( F_a(x) \) and \( F_e(x) \) to be of the form (2.5), for which \( f(k) = i/k \) for \( k \neq 0 \). For each \( k \), we begin by obtaining the exact roots \( z_n \) of (2.8). We then exactly invert (2.10) for the \( A^a_n \) or (2.11) for the \( A^e_n \). Summing these up as in (2.9) then yields the exact solution, which should indeed exhibit the features deduced from the asymptotic analysis. Of course, to actually sum up the series (2.9), we must truncate it at some finite value of \( k \). We chose to include all \( k \) up to \( \pm 1000 \). The forcing functions \( F(x) \) thus exhibit structure on scales as fine as \( O(1/1000) \), considerably finer than \( E^{1/3} \) for the
values of $E$ we will consider here. The fluid will thus indeed effectively see a discontinuity in the forcing functions, and will respond accordingly, giving us the true shear layer structure.

Figure 2 shows a horizontal slice through the shear layer induced by $F_s(x)$. Shown on the left is $u$, and on the right is $E^{-1/2}\psi$, as functions of $x$, for a fixed $z = 1/2$. From top to bottom, the Ekman number $E$ varies from $10^{-3}$ to $10^{-6}$. One notes first that away from the shear layer we do indeed have $u \approx F_s(x)$, as in (2.16a). The jump in $u$ is then accommodated by an $E^{1/4}$ shear layer, which also induces $E^{1/4}$ and $E^{1/3}$ shear layers in $\psi$. In the outer $E^{1/4}$ layer the flow is upward; in the inner $E^{1/3}$ layer the return flow is downward. (The values of $E$ used here are admittedly not small enough to distinguish clearly between $E^{1/4}$ and $E^{1/3}$, but for illustrative purposes they will suffice.) Finally, Stewartson has further demonstrated that within the shear layer $\psi = \mathcal{O}(E^{1/2})$; that is

![Figure 2](image_url)
why we plotted $E^{-1/2} \psi$ instead of just $\psi$. Away from the shear layer we do indeed have $\psi = O(E)$, as in (2.16b).

Figure 3 shows a horizontal slice through the shear layer induced by $F_{\phi}(x)$. Shown on the left is $E^{-1/6} \nu$, and on the right is $E^{-1/2} \psi$, again as functions of $x$, for a fixed $z = 1/2$. From top to bottom, $E$ again varies from $10^{-3}$ to $10^{-6}$. One notes first that away from the shear layer we do indeed have $\psi \approx 1/2E^{1/2}F_{\phi}(x)$, as in (2.18b). The jump in $E^{-1/2} \psi$ is then accommodated by an $E^{1/3}$ shear layer, which transfers fluid from one Ekman layer to the other, and which also induces another $E^{1/3}$ shear layer in $v$. (Because there is now only one shear layer thickness, it is indeed essentially exactly $E^{4/3}$ even for these values of $E$.) Again, Stewartson has further demonstrated that within the shear layer $v = O(E^{1/6})$; that is why we plotted $E^{-1/6} \nu$ instead of just $\nu$. Away from the shear layer we do indeed have $\nu = O(E^{3/2})$, as in (2.18a).

Figure 3 A horizontal slice through the shear layer induced by $F_{\phi}(x)$. On the left is $E^{-1/6} \nu$, on the right is $E^{-1/2} \psi$, as functions of $x$, for a fixed $z = 1/2$. From top to bottom $E$ varies from $10^{-3}$ to $10^{-6}$.
So, to summarize this section, we have demonstrated that the Cartesian geometry of Figure 1 yields exactly the same shear layers as Stewartson’s (1957) cylindrical geometry. The most important feature of these shear layers is that all structure in the forcing functions finer than $O(E^{1/3})$ is suppressed. In the next section we will show that a sufficiently strong magnetic field will suppress even this structure.

3. MAGNETIC SHEAR LAYERS

Basic Equations

The coupled momentum and induction equations are now,

\[ 2 \mathbf{k} \times \mathbf{U} = -\nabla p + E \nabla^2 \mathbf{U} + \Lambda (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{3.1a} \]

\[ \frac{\partial}{\partial t} \mathbf{B} = R_m^{-1} \nabla^2 \mathbf{B} + \nabla \times (\mathbf{U} \times \mathbf{B}), \tag{3.1b} \]

where the Elsasser number $\Lambda$ is a measure of the strength of the magnetic field $\mathbf{B}$, and the magnetic Reynolds number $R_m$ is a measure of the inductive versus the diffusive effects in (3.1b).

We then consider the limit of small $R_m$, and expand the field as

\[ \mathbf{B} = \mathbf{B}_0 + R_m \mathbf{b} + \ldots, \tag{3.2} \]

where $\mathbf{B}_0$ is the basic imposed field, and $\mathbf{b}$ is the induced field. Linearizing (3.1) then yields

\[ 2 \mathbf{k} \times \mathbf{U} = -\nabla p + E \nabla^2 \mathbf{U} + \Lambda R_m (\nabla \times \mathbf{b}) \times \mathbf{B}_0, \tag{3.3a} \]

\[ \frac{\partial}{\partial t} \mathbf{b} = \nabla^2 \mathbf{b} + \nabla \times (\mathbf{U} \times \mathbf{B}_0), \tag{3.3b} \]

where we are taking $\mathbf{B}_0$ to satisfy $\nabla \times \mathbf{B}_0 = 0$. It has been noted by Hollerbach (1994) that the solutions of the linearized equations (3.3) closely match the solutions of the full equations (3.1) even for relatively large $R_m$, a point we will return to in the conclusion. It is the asymptotics of these linearized equations that we will consider in this section.

Before proceeding further, we must decide what to take for the imposed field $\mathbf{B}_0$. It is here that the difference between Stewartson’s cylindrical geometry and our Cartesian geometry becomes crucial: In order for expansions of the form (2.6) to properly decouple all the modes, the field $\mathbf{B}_0$ must be uniform. But in cylindrical geometry, the only possible uniform field would be the vertical field $\mathbf{B}_0 = \mathbf{k}$. The effect of such a field on sidewall shear layers has already been considered by Ingham (1969), and has been found to be rather insignificant for all but the strongest fields. Far more interesting would be to take a field that has a component perpendicular to the shear layers. The
magnetic tension in the field lines would then be expected to have the greatest effect on the shear layer structure. As indicated above, in cylindrical geometry such a field presents a problem, but in Cartesian geometry it does not, and that is why we choose to work in the latter rather than the former. So, in summary, for the imposed field we take $B_0 = \hat{i}$, directly perpendicular to the shear layers.

By symmetry, the entire solution will then still be independent of $y$, and so we now use $\nabla \cdot U = \nabla \cdot b = 0$ to decompose

$$U = \nabla \times (\psi \hat{j}) + v \hat{j},$$

$$b = \nabla \times (a \hat{j}) + b \hat{j}.$$  \hspace{1cm} (3.4a, 3.4b)

For the choice $B_0 = \hat{i}$, the momentum equation (3.3a) then becomes

$$2 \frac{\partial}{\partial z} \psi + E \nabla^2 v = -\tilde{\Lambda} \frac{\partial}{\partial x} b,$$  \hspace{1cm} (3.5a)

$$2 \frac{\partial}{\partial z} v - E \nabla^4 \psi = \tilde{\Lambda} \frac{\partial}{\partial x} \nabla^2 a,$$  \hspace{1cm} (3.5b)

where $\tilde{\Lambda} \equiv \Lambda R_m$ is a modified Elsasser number. Similarly, the steady-state version of the induction equation (3.3b) then becomes

$$0 = \nabla^2 b + \frac{\partial}{\partial x} v,$$  \hspace{1cm} (3.6a)

$$0 = \nabla^2 a + \frac{\partial}{\partial x} \psi.$$  \hspace{1cm} (3.6b)

For the flow, the associated boundary conditions at $z = \pm 1$ remain as before,

$$v = F_s(x) \pm F_d(x),$$  \hspace{1cm} (3.7a)

$$\psi = \frac{\partial}{\partial z} \psi = 0.$$  \hspace{1cm} (3.7b)

For the field, the boundary conditions become

$$b = 0,$$  \hspace{1cm} (3.8a)

$$\frac{\partial}{\partial z} a \pm ka = 0,$$  \hspace{1cm} (3.8b)

taking the external regions $|z| > 1$ to be insulators. In fact, only the boundary condition on $b$ will actually be necessary. Because $a$ only appears as $\nabla^2 a$, the boundary condition on $a$ would only be necessary if we subsequently wished to invert $\nabla^2 a$ to obtain $a$. It is
these equations (3.5) and (3.6), and the associated boundary conditions (3.7) and (3.8), that we wish to solve. In particular, we will show that for \( \tilde{A} \gtrsim O(E^{1/3}) \) the magnetic field will suppress the structure of the previously obtained shear layers.

**Exact Solution**

Extending our previous analysis, the general solution is now

\[
v = \sum_{k = -\infty}^{\infty} \sum_{n = 1}^{4} \left[ A_n^v(k)e^{ikx}\cosh(\alpha_nz) + A_n^v(k)e^{ikx}\sinh(\alpha_nz) \right], \tag{3.9a}
\]

\[
\psi = \sum_{k = -\infty}^{\infty} \sum_{n = 1}^{4} \left[ B_n^\psi(k)e^{ikx}\sinh(\alpha_nz) + B_n^\psi(k)e^{ikx}\cosh(\alpha_nz) \right], \tag{3.9b}
\]

\[
b = \sum_{k = -\infty}^{\infty} \sum_{n = 1}^{4} \left[ C_n^b(k)e^{ikx}\cosh(\alpha_nz) + C_n^b(k)e^{ikx}\sinh(\alpha_nz) \right], \tag{3.9c}
\]

where

\[
B_n = \frac{E(\alpha_n^2 - k^2)^2 + \tilde{A}k^2}{2\alpha_n(\alpha_n^2 - k^2)} A_n, \quad C_n = -\frac{ik}{\alpha_n^2 - k^2} A_n, \tag{3.9d,e}
\]

and we have used (3.6b) to eliminate \( \nabla^2 v \) in (3.5b). These expansions will satisfy (3.5) and (3.6) if \( \alpha \) now satisfies the equation

\[
[E(\alpha^2 - k^2)^2 + \tilde{A}k^2]^2 = -4\alpha^2(\alpha^2 - k^2), \tag{3.10}
\]

and we note that this quartic in \( \alpha^2 \) does indeed have four roots \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), with positive real parts, as indicated in (3.9).

For each Fourier mode \( k \), the boundary conditions (3.7a,b) and (3.8a) then yield

\[
\sum_{n = 1}^{4} \cosh(\alpha_n)A_n^v(k) = f_1(k), \tag{3.11a}
\]

\[
\sum_{n = 1}^{4} \frac{E(\alpha_n^2 - k^2)^2 + \tilde{A}k^2}{\alpha_n(\alpha_n^2 - k^2)} \sinh(\alpha_n)A_n^\psi(k) = 0, \tag{3.11b}
\]

\[
\sum_{n = 1}^{4} \frac{E(\alpha_n^2 - k^2)^2 + \tilde{A}k^2}{\alpha_n^2 - k^2} \cosh(\alpha_n)A_n^b(k) = 0, \tag{3.11c}
\]

\[
\sum_{n = 1}^{4} \frac{k}{\alpha_n^2 - k^2} \cosh(\alpha_n)A_n^b(k) = 0, \tag{3.11d}
\]
and

\[ \sum_{n=1}^{4} \sinh(\alpha_n) A_n^\alpha(k) = f_\delta(k), \]  

(3.12a)

\[ \sum_{n=1}^{4} \frac{E(\alpha_n^2 - k^2)^2 + \tilde{\Lambda}k^2}{\alpha_n(\alpha_n^2 - k^2)} \cosh(\alpha_n) A_n^\alpha(k) = 0, \]  

(3.12b)

\[ \sum_{n=1}^{4} \frac{E(\alpha_n^2 - k^2)^2 + \tilde{\Lambda}k^2}{\alpha_n^2 - k^2} \sinh(\alpha_n) A_n^\alpha(k) = 0, \]  

(3.12c)

\[ \sum_{n=1}^{4} \frac{k}{\alpha_n^2 - k^2} \sinh(\alpha_n) A_n^\alpha(k) = 0. \]  

(3.12d)

For each \( k \), the four equations (3.11) determine the four symmetric coefficients \( A_n^\alpha(k) \), and the four equations (3.12) determine the four antisymmetric coefficients \( A_n^\beta(k) \), again yielding an exact formal solution.

**Asymptotic Analysis**

Again, we wish to consider the asymptotic behaviour of this exact solution in the limit of small \( E \). The roots of the characteristic equation (3.10) are then

\[ \alpha_1 \approx Ek^{3/2} + \tilde{\Lambda}k/2, \quad \alpha_2 \approx E^{-1/2}(1 + i)(1 + \tilde{\Lambda}Ek^2/8), \]  

(3.13a,b)

\[ \alpha_3 \approx E^{-1/2}(1 - i)(1 + \tilde{\Lambda}Ek^2/8), \quad \alpha_4 \approx k(1 - \tilde{\Lambda}^2/8), \]  

(3.13c,d)

again valid provided that \( k \ll E^{-1/2} \), and also \( \tilde{\Lambda} \ll 1 \). This might seem peculiar, restricting myself to very weak fields, but it will turn out that \( \tilde{\Lambda} \ll O(1) \) is more than sufficient to suppress the previously obtained shear layer structure.

As one might expect, it is primarily the change in \( \alpha_1 \) that will have a profound influence on the structure of the shear layers. In contrast, the Ekman layer modes associated with \( \alpha_2 \) and \( \alpha_3 \) are essentially unchanged. As with the shear layers, the Ekman layers are not much affected by a magnetic field that is not perpendicular to them. Finally, the new root \( \alpha_4 \) will turn out not to be associated with any boundary layer structure at all; it is instead associated primarily with the interior structure of \( b \).

Having obtained these approximate roots (3.13), the boundary conditions (3.11) and (3.12) now yield the approximate coefficients

\[ A_1^\alpha(k) \approx k f_\delta(k)/[2E^{-1/2} \sinh(\alpha_1) + (k - \tilde{\Lambda}E^{-1/2} \tanh \alpha_4) \cosh(\alpha_1)], \]  

(3.14a)

\[ A_1^\beta(k) \approx k f_\delta(k)/[2E^{-1/2} \cosh(\alpha_1) + (k - \tilde{\Lambda}E^{-1/2} \coth \alpha_4) \sinh(\alpha_1)], \]  

(3.14b)

much the same as before. However, because \( \alpha_4 \) is now different, the shear layer structure we deduce from (3.14) will also be quite different.
Most importantly, we will now have $\alpha_k \gtrsim O(1)$ for all $k$ greater than the lesser of $E^{-1/3}$ and $\tilde{\Lambda}^{-1}$. That is, where before all structure on scales finer than $E^{1/3}$ was suppressed, if $\tilde{\Lambda} \gtrsim E^{1/3}$, all structure on scales finer than $\tilde{\Lambda}$ will be suppressed. Thus, if $\tilde{\Lambda}$ is increased beyond $O(E^{1/3})$ toward $O(1)$, the shear layers will gradually thicken, until they fill the whole region, and all structure is suppressed. This suppression of the shear layers even for such relatively weak fields as $\tilde{\Lambda} \gtrsim E^{1/3}$ is probably the most important result to emerge from this work.

We turn now to a more detailed analysis of the scalings in the denominators of (3.14), to obtain the precise structure of these shear layers. In the denominator of (3.14b), for \( \tilde{\Lambda} \gtrsim E^{1/3} \) there are still only two possible scalings: for $k \lesssim \tilde{\Lambda}^{-1}$ it is essentially constant, and for $k \gtrsim \tilde{\Lambda}^{-1}$ it is exponentially large. Thus, for $\tilde{\Lambda} \gtrsim E^{1/3}$ the $E^{1/3}$ layer simply becomes the $\tilde{\Lambda}$ layer indicated above. For the antisymmetric forcing, there is again no other structure. In contrast, in the denominator of (3.14a), for $E^{1/2} \lesssim \tilde{\Lambda} \lesssim E^{1/3}$ there are still three possible scalings: for $k \lesssim \tilde{\Lambda}^{1/2} E^{-1/2}$ it is linear in $k$, for $\tilde{\Lambda}^{1/2} E^{-1/2} \lesssim k \lesssim E^{-1/3}$ it is cubic in $k$, and for $k \gtrsim E^{-1/3}$ it is again exponentially large, whereas for $\tilde{\Lambda} \gtrsim E^{1/3}$ there are only two possible scalings: for $k \lesssim \tilde{\Lambda}^{-1}$ it is linear in $k$, and for $k \gtrsim \tilde{\Lambda}^{-1}$ it is exponentially large. Thus, for $\tilde{\Lambda} \gtrsim E^{1/2}$ the $E^{1/4}$ layer becomes a $\tilde{\Lambda}^{-1/2} E^{1/2}$ layer, that is, it becomes thinner, not thicker, until for $\tilde{\Lambda} \approx E^{1/3}$ it merges with the $E^{1/3}$ layer. For $\tilde{\Lambda} \gtrsim E^{1/3}$ this $E^{1/3}$ layer then becomes the $\tilde{\Lambda}$ layer indicated above.

To summarize, the antisymmetric shear layer structure is unaffected for $\tilde{\Lambda} \lesssim E^{1/3}$. For $\tilde{\Lambda} \gtrsim E^{1/3}$ the $E^{1/3}$ layer then becomes a thicker $\tilde{\Lambda}$ layer. In contrast, the symmetric shear layer structure is unaffected only for $\tilde{\Lambda} \lesssim E^{1/2}$. For $\tilde{\Lambda} \gtrsim E^{1/2}$ the $E^{1/4}$ layer then becomes a thinner $\lambda^{1/2} E^{1/2}$ layer, which merges with the $E^{1/3}$ layer when $\lambda \approx E^{1/3}$. For $\tilde{\Lambda} \gtrsim E^{1/3}$ the combined $E^{1/3}$ layer then becomes a thicker $\tilde{\Lambda}$ layer, as before. That the $E^{1/3}$ layers should become thicker for $\tilde{\Lambda} \gtrsim E^{1/3}$ is not surprising; it is essentially what one would expect from the "stiffening" effect that the imposed field has on the fluid. That the $E^{1/4}$ layer should become thinner for $E^{1/2} \lesssim \tilde{\Lambda} \lesssim E^{1/3}$ was, to me at least, quite surprising, as it would seem to imply that the shear increases, rather than decreases. To demonstrate that this is not in fact the case, we now consider not just the thicknesses of the various layers, but also the magnitudes of the jumps in the various quantities across them.

First, according to (2.16a), the primary purpose of the symmetric shear layer is to accommodate the $O(1)$ jump in $v$. Within the $E^{1/4}$ layer, the shear $\partial v/\partial x$ is then $O(E^{-1/4})$. For $\tilde{\Lambda} \gtrsim E^{1/2}$ one can similarly show that the jump in $v$ is only $O(\tilde{\Lambda}^{-1} E^{1/2})$. Within the $\tilde{\Lambda}^{1/2} E^{1/2}$ layer, the shear $\partial v/\partial x$ is then only $O(\tilde{\Lambda}^{-1/2})$. Although the shear layer becomes thinner, the shear within it is thus nevertheless reduced, as one would expect. Next, according to (2.18b), the primary purpose of the antisymmetric shear layer is to accommodate the $O(E^{1/2})$ jump in $\psi$. One can easily show that the jump in $\psi$ always remains $O(E^{1/2})$. As this shear layer then becomes thicker, the same jump is accommodated across a greater thickness, again reducing the shear.

Finally, more for the sake of completeness than anything else, a few words are in order on the mode associated with the new root $\alpha_4$. As in (3.14), one now obtains the approximate coefficients

$$A_4^2(k) \approx \frac{\tilde{\Lambda}^2}{4} \cosh(x_1) A_4^2(k)/\cosh(x_4),$$

(3.15a)
\[ A_2^\alpha(k) \approx \frac{\hat{\Lambda}^2}{4} \sinh(\alpha_4) A_1^\alpha(k)/\sinh(\alpha_4). \]  

(3.15b)

Because \( \alpha_4 \geq O(1) \) for all \( k \), this mode is only important for the very smallest \( k \), that is, this is an interior mode rather than a shear layer mode. In any case, because of the \( \hat{\Lambda}^2 \) factor, it is utterly negligible as far as the flow structure is concerned. It is instead associated primarily with the structure of the field \( b \); from (3.9e) and (3.13d) one obtains

\[ C_4^\alpha(k) \approx \frac{i}{k} \cosh(\alpha_4) A_1^\alpha(k)/\cosh(\alpha_4), \]  

(3.16a)

\[ C_4^\alpha(k) \approx \frac{i}{k} \sinh(\alpha_4) A_1^\alpha(k)/\sinh(\alpha_4). \]  

(3.16b)

The detailed structure of \( b \) will not be considered further, however.

**Examples**

Again, a few examples will serve to illustrate the structure of these magnetic shear layers. We will consider only the “strong” field regime \( \hat{\Lambda} \geq O(E^{-1/3}) \), where the thickness of the shear layers is determined entirely by \( \hat{\Lambda} \). Figures 4 and 5 show horizontal slices through the shear layers induced by \( F_3(x) \) and \( F_4(x) \), respectively. Shown on the left is \( \hat{\Lambda}E^{-1/2}v \), and on the right is \( E^{1/2}\psi \), as functions of \( x \), for a fixed \( z = 1/2 \). In each case, the top two rows are for \( E = 10^{-6} \), the bottom two rows for \( E = 10^{-12} \). For each \( E \), the upper row is for \( \hat{\Lambda} = 0.05 \), the lower row for \( \hat{\Lambda} = 0.5 \). As expected, in both Figures 4 and 5 the thickness of the shear layers is determined entirely by \( \hat{\Lambda} \), increasing with increasing \( \hat{\Lambda} \). Indeed, the Ekman number \( E \) enters into the solutions only as this overall factor of \( E^{1/2} \); the solutions for \( E \) equals \( 10^{-6} \) and \( 10^{-12} \) are otherwise virtually indistinguishable.

Turning now to some of the details of the structure, for the symmetric shear layer in Figure 4, the jump in \( v \) indeed decreases from \( O(1) \) to \( O(\hat{\Lambda}^{-1} E^{1/2}) \), as indicated above. Incidentally, one notes also that away from the shear layer \( \psi \) increases from \( O(E) \) to \( O(E^{1/2}) \). This is readily understood in terms of the disappearance of the \( E^{1/4} \) layer; the upward flow that was accommodated in that layer is instead accommodated throughout the whole of the interior. For the antisymmetric shear layer in Figure 5, the jump in \( \psi \) indeed remains constant at \( O(E^{1/2}) \), again as indicated above. One notes also that within the shear layer \( v \) decreases from \( O(E^{1/6}) \) to \( O(\hat{\Lambda}^{-1} E^{1/2}) \). This is readily understood in terms of the transition from an \( E^{1/3} \) to a \( \hat{\Lambda} \) layer: an \( O(E^{1/2}) \) jump in \( \psi \) across an \( E^{1/3} \) layer implies a vertical flow \( \partial \psi/\partial z = O(E^{1/6}) \); an \( O(E^{1/2}) \) jump in \( \psi \) across a \( \hat{\Lambda} \) layer implies a vertical flow \( \partial \psi/\partial z = O(\hat{\Lambda}^{-1} E^{1/2}) \). It is then not surprising that in each case this vertical flow \( \partial \psi/\partial z \) should induce a zonal flow \( v \) of the same magnitude.
4. CONCLUSION

In this work we have considered the effect of imposing a magnetic field across shear layers in a rapidly rotating plane layer. In the non-magnetic regime, the purpose of the shear layer induced by the symmetric forcing $F_s(x)$ is to accommodate an $O(1)$ jump in $v$, and the purpose of the shear layer induced by the antisymmetric forcing $F_a(x)$ is to accommodate an $O(E^{1/2})$ jump in $\psi$, just as in Stewartson's (1957) cylindrical geometry. The jump in $v$ is across an $E^{1/4}$ layer, which also induces $E^{1/4}$ and $E^{1/3}$ layers in $\psi$. The jump in $\psi$ is across an $E^{1/3}$ layer, which also induces another $E^{1/3}$ layer in $v$. In the magnetic regime, then, there is first a transition regime $E^{1/2} \lesssim \Lambda \lesssim E^{1/3}$ in which the
symmetric $E_{1/4}$ layer becomes a thinner $\tilde{\Lambda}^{-1/2}E^{1/2}$ layer. Simultaneously, however, the $O(1)$ jump in $v$ becomes a smaller $O(\tilde{\Lambda}^{-1}E^{1/2})$ jump, so that the shear $\partial v/\partial x$ is in fact reduced even in this regime. Both the symmetric and antisymmetric $E_{1/3}$ layers are unaffected in this regime. There is next the true strong field regime $\tilde{\Lambda} \gtrsim E^{1/3}$, in which both $E_{1/3}$ layers become thicker $\tilde{\Lambda}$ layers. The jump in $v$ is still $O(\tilde{\Lambda}^{-1}E^{1/2})$, and the jump in $\psi$ is still $O(E^{1/2})$. For any fixed $\tilde{\Lambda} \gtrsim E^{1/3}$, all three components of the flow are thus $O(E^{1/2})$; that is, essentially the entire adjustment to the imposed forcing occurs in the Ekman layers (which we have not considered in detail here).

Returning now to the proper spherical geometry, we note that these asymptotic results are in exact agreement with the numerical results previously obtained by

![Figure 5](image)

**Figure 5** A horizontal slice through the magnetic shear layer induced by $F(x)$. On the left is $\tilde{\Lambda}E^{-1/2}v$, on the right is $E^{-1/2}\psi$. For the top two rows $E = 10^{-6}$; for the bottom two rows $E = 10^{-12}$. For each $E$, the upper row is for $\tilde{\Lambda} = 0.05$, the lower for $\tilde{\Lambda} = 0.5$. 


Hollerbach (1994), which indicated that for a sufficiently strong field the shear layers are completely suppressed, with the entire adjustment occurring in the Ekman layers. The main difference between the plane layer and the spherical geometries is probably that in the spherical case there is, in addition to the $E^{1/4}$ and $E^{1/3}$ layers, an intermediate $E^{2/7}$ layer (Stewartson, 1966). One must confess that the analysis presented here says nothing about the adjustment of this layer. Nevertheless, having demonstrated that both the thicker $E^{1/4}$ and thinner $E^{1/3}$ layers are ultimately suppressed, it seems likely that this intermediate $E^{2/7}$ layer is then also suppressed. Thus, it would seem that $A \gtrsim O(E^{1/3})$ constitutes the strong field regime even in the proper spherical geometry.

Finally, we should comment briefly on the assumption of small $R_m$, which was crucial in obtaining the linearized equations (3.3). A rigorous asymptotic analysis of the nonlinear equations (3.1), valid for any $R_m$, is of course beyond the scope of this work. Nevertheless, we can give a qualitative argument for why the analysis presented here might be expected to be valid for surprisingly large $R_m$: When we introduced the magnetic Reynolds number $R_m$, we stated that it is “a measure of the inductive versus the diffusive effects.” Implicit in that statement is the assumption that the fluid flow in (3.1b) is $O(1)$. However, we saw that, away from the Ekman layers at least, the flow is in fact only $O(E^{-1/2})$. Thus, a more proper measure of the inductive versus the diffusive effects might be $R_m E^{1/2}$, and this will be small not only for the range $R_m \lesssim O(1)$, but for the rather more generous range $R_m \lesssim O(E^{-1/2})$. This would certainly agree with the numerical results of Hollerbach (1994), where for $E = 10^{-4}$ virtually no differences were obtained for $R_m \lesssim O(100)$ between the solutions of the nonlinear equations (3.1) and the solutions of the linearized equations (3.3). This in turn would suggest that the analysis presented here is indeed valid in the Earth’s core, where the magnetic Reynolds number is $O(100)$, but the Ekman number is $O(10^{-12})$.

Acknowledgments

I thank Dr. Ruzmaikin for various discussions on magnetohydrodynamic shear layers, and for sending me a preprint of the paper by Kleeorin et al.

References


