Newton-Equivalent Hamiltonians for the Harmonic Oscillator

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We consider a one-parameter family of Hamilton functions yielding the Newton equation of the harmonic oscillator, \( \ddot{x} + \omega^2 x = 0 \). The parameter may be viewed as the speed of light \( c \), the nonrelativistic limit \( c \to \infty \) yielding the usual Hamiltonian. For \( c \ll 1 \), the classical Hamiltonians are the product of a function of \( x \) and a function of \( p \). In the quantum case, with a suitable ordering, we explicitly find the spectrum and the eigenfunctions of the Hamiltonian.

1. INTRODUCTION

As motivation for the present work, consider a classical mechanical system with one degree of freedom, whose motion is given by the Newton equation

\[
m \frac{d^2}{dt^2} x + \frac{d}{dx} V(x) = 0. \tag{1.1}
\]

The integration of (1.1) provides the time dependence of the coordinate \( x(t) \) of a mass-\( m \) particle in the potential \( V(x) \). We recall that this problem admits a Langrangian and a Hamiltonian reformulation. In the first setting, one obtains (1.1) via the Euler–Lagrange derivative of a function \( L = L(x, y) \),

\[
d \frac{\partial L}{\partial y} - \frac{\partial L}{\partial x} = 0, \quad \frac{dx}{dt} = y. \tag{1.2}
\]

In the second, one rewrites the second order differential equation (1.1) as a system of two coupled first order equations via a function \( H = H(x, p) \),

\[
\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \tag{1.3}
\]
Though these two approaches to the problem of motion are elegant and powerful, they both lead to the problem of solving the Newton equation (1.1). On the other hand, assuming that Lagrange and Hamilton functions yielding a given Newton equation exist, they need not be unique. (The existence problem was posed more than a century ago by Helmholtz, and is now a well-established topic in classical mechanics, known as the inverse problem; see, for instance, the review Ref. [1].)

In the following, two different Lagrange or Hamilton functions will be termed Newton-equivalent if they yield the same Newton equation. As we show in Section 2, a quite simple 1-parameter family of Newton-equivalent Hamiltonians for (1.1) exists. For a larger class of equations of motion with velocity- and time-dependent forces this equivalence (termed q-equivalence, with q denoting position) has been investigated in Refs. [2, 3].

Consider now the quantum mechanical problem associated with the Newton equation (1.1), this being our main concern here. Let us recall that in the quantum context the Lagrangian or Hamiltonian plays a far more fundamental role. Indeed, to go from classical to quantum mechanics one chooses either the Lagrange function or the Hamilton function as a starting point. For the first choice one then employs path integrals, a quantization procedure that we will not be concerned with here.

Turning to the second choice (“canonical quantization”), one reinterprets the Hamilton function \( H(x, p) \) as a function of operators \( \hat{x} \) and \( \hat{p} \) satisfying the Heisenberg relation

\[
[\hat{x}, \hat{p}] = i\hbar.
\]  

(1.4)

This prescription may give rise to ambiguities in \( H(\hat{x}, \hat{p}) \) due to different orderings, even when one insists on formal self-adjointness (as one should).

Having recalled these well-known facts, let us return to our 1-parameter family of Hamilton functions yielding (1.1). The customary choice,

\[
H_E(x, p) = \frac{1}{2m} p^2 + V(x),
\]  

(1.5)

does not lead to ordering ambiguities at the quantum level. Now this choice is present as a limit case in our setting. Therefore, it is natural to ask in what way—if any—the quantum characteristics, for instance the spectrum of the Hamilton operator, depend on the choice of the parameter. If we make the harmonic oscillator choice

\[
V(x) = \frac{1}{2} m \omega^2 x^2
\]  

(1.6)

in the Newton equation (1.1), our results concerning the quantum mechanics of the associated 1-parameter family yield an explicit answer to this question.

The plan of this paper is as follows. In Section 2 we detail the above-mentioned 1-parameter family of Newton-equivalent Hamilton functions yielding (1.1). We also indicate in what sense the parameter may be interpreted as the speed of light.

In Section 3 we specialize to a 1-parameter family of quantum Hamiltonians corresponding to the oscillator choice (1.6). This family arises by fixing the ordering of noncommuting factors in a certain way, which enables us to solve the diagonalization problem completely in terms of orthogonal polynomials. The spectrum we find has the same characteristics as the well-known spectrum associated to the Hamiltonian

\[
\hat{H}_E = \hat{p}^2 / 2m + m \omega^2 \hat{x}^2 / 2.
\]  

(1.7)

but for one striking difference: The ground state energy equals the minimum of the classical
Hamiltonian function. (In quantum field theoretic parlance, our generalized oscillator has vanishing vacuum energy.)

In Section 4 we elaborate on the findings of Section 3, generalizing some standard algebraic features associated with $\hat{H}_E$ (1.7) and its classical version. In particular, we introduce and exploit raising and lowering operators. Moreover, we establish the relation of our eigenfunction basis to a special type of Meixner–Pollaczek polynomials.

In Section 5 we compare our results with previous results in the literature concerning generalized harmonic oscillators. In particular, we mention various notions of “relativistic” harmonic oscillator we have come across, observing that the notion arising from the present perspective has not been proposed before.

2. THE 1-PARAMETER FAMILY AND ITS RELATIVISTIC INTERPRETATION

When we take the time derivative of the first equation in (1.3) and use the second one, we see that a Hamilton function $H(x, p)$ entails the Newton equation (1.1) if and only if it satisfies

$$\frac{\partial^2 H}{\partial x \partial p} \frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2} \frac{\partial H}{\partial x} + \frac{1}{\hbar} \frac{\partial V}{\partial x} = 0. \quad (2.1)$$

Therefore, for a given potential function $V(x)$, two Hamilton functions are Newton-equivalent if they both satisfy (2.1).

Though we are not concerned here with the general solution of (2.1), it is illuminating to see how the 1-parameter family (2.8) at issue in this paper arises when we assume a certain dependence of $H$ on $x$ and $p$. Specifically, let us study the two cases where $H$ has a separable dependence on $x$ and $p$ that is additive or multiplicative.

In the first case $H$ reads

$$H(x, p) = F(p) + G(x). \quad (2.2)$$

Then (2.1) becomes

$$-F''(p)G'(x) + \frac{1}{\hbar} V'(x) = 0. \quad (2.3)$$

Therefore, $F(p)$ is of the form $Ap^2 + Bp + C$, with $B$ and $C$ arbitrary. Hence we obtain $G(x) = V(x)/(2\hbar) + D$. The upshot is that (2.2) leads to the standard choice $H_E(x, p) (1.5)$ up to the ambiguities due to the arbitrary parameters $A$, $B$, $C$, and $D$.

In the second case we have

$$H(x, p) = F(p)G(x) \quad (2.4)$$

and (2.1) yields

$$[F'(p)^2 - F''(p)F(p)]G'(x)G(x) + \frac{1}{\hbar} V'(x) = 0. \quad (2.5)$$

Therefore, $F(p)$ satisfies the equation

$$F'(p)^2 - F''(p)F(p) = -A. \quad (2.6)$$
For positive $A$ this second order ODE is solved by $F(p) = C_1 \cosh(C_2 p + C_3)$ with $C_1^2 C_2^2 = A$, for negative $A$ by $F(p) = C_1 \sinh(C_2 p + C_3)$ with $C_1^2 C_2^2 = -A$. In either case, we deduce

$$[G(x)]^2 = 2V(x)/(mA) + D. \quad (2.7)$$

Clearly, the rhs can only be positive when $V(x)$ is either bounded below or bounded above. Assuming $V(x)$ is bounded below, we can choose $A$ positive and then choose $D$ such that the rhs of (2.7) is positive. In this way we obtain a family of positive Hamiltonians of the product form (2.4).

The analysis leading from (2.4) to (2.7) shows in particular that all of the Hamiltonians in the 1-parameter family

$$H_c(x, p) = 4mc^2 \cosh \left( \frac{p}{2mc} \right) \left( 1 + \frac{V(x)}{2mc^2} \right)^{1/2}, \quad c \in (0, \infty), \quad (2.8)$$

give rise to the Newton equation (1.1). (Of course, this can also be checked directly via (1.3).) Observe that one needs to require

$$V(x) > -2mc^2 \quad (2.9)$$

for the Hamiltonians (2.8) to be positive.

The 1-parameter family (2.8) includes $H_E(x, p)$ (1.5) as a limit case since we have

$$\lim_{c \to \infty} \left[ H_c(x, p) - 4mc^2 \right] = H_E(x, p). \quad (2.10)$$

We now explain in what sense this limit may be viewed as a nonrelativistic limit with the parameter $c$ playing the role of the speed of light. First, let us observe that we may reinterpret $H_E(x, p)$ as the center-of-mass Hamiltonian of a system of two particles with equal mass $\mu = 2m$ and Hamiltonian

$$H_{NR} = \frac{1}{2\mu} p_1^2 + \frac{1}{2\mu} p_2^2 + V(x_1 - x_2), \quad \mu = 2m. \quad (2.11)$$

Indeed, in center-of-mass coordinates

$$P = p_1 + p_2, \quad X = (x_1 + x_2)/2, \quad p = (p_1 - p_2)/2, \quad x = x_1 - x_2, \quad (2.12)$$

we obtain

$$H_{NR} = \frac{p^2}{4\mu} + H_E(x, p). \quad (2.13)$$

(Here $2\mu = 4m$ is the total mass and $\mu/2 = m$ is the reduced mass.)

Second, we recall that a Hamiltonian of the form (2.11) can be viewed as the time translation generator of a representation of the Galilei group. Indeed, with the space translation generator given by the total momentum

$$P_{NR} = p_1 + p_2 = P, \quad (2.14)$$
and the Galilei boost generator given by

\[ B = \mu(x_1 + x_2) = 2\mu X, \]  

one obtains a representation of the Lie algebra of the Galilei group:

\[ \{ H_{NR}, P_{NR} \} = 0, \quad \{ B, H_{NR} \} = P_{NR}, \quad \{ B, P_{NR} \} = 2\mu. \]  

(Note that the constant 2\mu generates a trivial flow.)

Third, consider the functions

\[ H_R = \mu c^2 \left[ \cosh\left( \frac{p_1}{\mu c} \right) + \cosh\left( \frac{p_2}{\mu c} \right) \right] \left( 1 + \frac{V(x_1 - x_2)}{\mu c^2} \right)^{1/2}, \]  

\[ P_R = \mu c \left[ \sinh\left( \frac{p_1}{\mu c} \right) + \sinh\left( \frac{p_2}{\mu c} \right) \right] \left( 1 + \frac{V(x_1 - x_2)}{\mu c^2} \right)^{1/2}. \]  

Together with B, they represent the Lie algebra of the Poincaré (inhomogeneous Lorentz) group:

\[ \{ H_R, P_R \} = 0, \quad \{ B, H_R \} = P_R, \quad \{ B, P_R \} = H_R/c^2. \]  

Thus \( H_R \) (2.17) may be viewed as the Hamiltonian of a relativistically invariant system of two particles on the line.

Observe that we may rewrite \( H_R \) and \( P_R \) in terms of the center-of-mass variables (2.12) as

\[ H_R = 2\mu c^2 \cosh\left( \frac{P}{2\mu c} \right) \cosh\left( \frac{P}{\mu c} \right) \left( 1 + \frac{V(x)}{\mu c^2} \right)^{1/2}, \]  

\[ P_R = 2\mu c \sinh\left( \frac{P}{2\mu c} \right) \cosh\left( \frac{P}{\mu c} \right) \left( 1 + \frac{V(x)}{\mu c^2} \right)^{1/2}. \]  

Recalling (2.8), this can be abbreviated as

\[ H_R = \cosh\left( \frac{P}{2\mu c} \right) H_c(x, p), \]  

\[ P_R = \frac{1}{c} \sinh\left( \frac{P}{2\mu c} \right) H_c(x, p). \]  

Thus, \( H_c(x, p) \) may be interpreted as the center-of-mass Hamiltonian of a relativistically invariant 2-particle system.

3. THE QUANTUM CASE: SPECTRUM AND EIGENFUNCTIONS
FOR THE OSCILLATOR FAMILY

From now on we specialize to the harmonic oscillator potential (1.6). Using the parameter

\[ \beta = (2mc)^{-1}, \]
our family of Newton-equivalent Hamiltonians (2.8) can then be written

\[
H_c = H(\beta; x, p) = \frac{1}{\beta^2 m} \cosh(\beta p)(1 + \beta^2 m^2 \omega^2 x^2)^{1/2}.
\]  

(3.2)

Thus the canonical quantization prescription (cf. (1.4))

\[
p \to \hat{p} = -i\hbar \partial_x
\]  

(3.3)

gives rise to noncommuting factors.

Clearly, there are many ways to pull the factors apart before substituting (3.3), even when we insist on obtaining a formally self-adjoint and parity-invariant operator (as we do). We make a special choice that was inspired by previous work of one of us [4] and reconsider ordering ambiguities towards the end of this section. Our choice is given by

\[
\hat{H}(\beta) \equiv \frac{1}{2\beta^2 m^2} [(1 + i\beta m \omega \hat{x})^{1/2} \exp(-i\hbar \beta \partial_x)(1 - i\beta m \omega \hat{x})^{1/2} + (i \to -i)].
\]  

(3.4)

Here we view \( \hat{H}(\beta) \) at first as an analytic difference operator, acting on functions \( \Psi(x), x \in \mathbb{R} \), that are analytic in a strip \( |\text{Im} \, x| \leq \hbar \beta \), with possibly square-root branch points for \( x = \pm i(\beta m \omega)^{-1} \).

Thus the exponentials simply shift the argument of \( \Psi(x) \) by \( \pm i \hbar \beta \).

A crucial feature of this type of operator is that the associated Schrödinger equation

\[
\hat{H}(\beta) \Psi = E \Psi
\]  

(3.5)

admits an infinite-dimensional solution space: Assuming \( F(x) \) solves (3.5) and \( f(x) \) is a meromorphic function with period \( i\hbar \beta \), then also \( f(x) F(x) \) solves (3.5). This state of affairs is the main reason that analytic difference operators cannot be readily studied within the well-established Hilbert space framework applying to ordinary differential and discrete difference operators (as expounded for example in Ref. [5]). Indeed, from previous explicit examples [6] it transpires that even the “free” exponentials \( \exp(\pm ia \partial_x), a > 0 \), admit definitions as bona fide self-adjoint operators that are vastly different from their obvious definition via Fourier transformation. For the analytic difference operator (3.4) under consideration, unbounded functions of \( \hat{x} \) also occur, so that a direct definition of \( \hat{H}(\beta) \) as a symmetric operator on a dense subspace of \( L^2(\mathbb{R}, dx) \) appears quite problematic at face value.

Even so, there is a natural way around these snags for the simple case at hand. Specifically, we will construct eigenfunctions of the analytic difference operator \( \hat{H}(\beta) \) with positive eigenvalues, which not only belong to the Hilbert space \( L^2(\mathbb{R}, dx) \), but are also pairwise orthogonal and complete. This will enable us to associate to \( \hat{H}(\beta) \) a self-adjoint operator on \( L^2(\mathbb{R}, dx) \), denoted by the same symbol.

As a first step, consider the function

\[
\Psi_0^{(\beta)}(x) \equiv \left( \Gamma\left( \frac{1}{\hbar \beta^2 m \omega} + \frac{i x}{\hbar \beta} \right) \Gamma\left( \frac{1}{\hbar \beta^2 m \omega} - \frac{i x}{\hbar \beta} \right) \right)^{1/2}.
\]  

(3.6)

This function satisfies

\[
\hat{H}(\beta) \Psi_0^{(\beta)}(x) = \frac{1}{\beta^2 m} \Psi_0^{(\beta)}(x)
\]  

(3.7)
by virtue of the difference equation

\[ \Gamma(z + 1) = z \Gamma(z). \]  

(3.8)

(In the next section we will arrive at this eigenfunction in a systematic way.) Clearly \( \Psi_0^{(\beta)} \) is even and positive. Furthermore, on account of the well-known \( \Gamma \)-function asymptotics (Stirling formula), it has the asymptotic behavior

\[
\Psi_0^{(\beta)}(x)^2 \sim 2\pi (|x|/\hbar \beta)^{-1+2/\hbar \beta^2 m \omega} \exp(-\pi |x|/\hbar \beta), \quad x \to \pm \infty.
\]

(3.9)

Therefore, \( \Psi_0^{(\beta)}(x) \) is square-integrable as well.

Next, we make a similarity transformation to the Hilbert space

\[ \mathcal{H}_\beta \equiv L^2(R, \Psi_0^{(\beta)}(x)^2 \, dx). \]  

(3.10)

Thus, we calculate

\[
\hat{H}(\beta) \rightarrow K_\beta \equiv \Psi_0^{(\beta)}(\hat{x})^{-1} \hat{H}(\beta) \Psi_0^{(\beta)}(\hat{x})
\]

\[
= \frac{1}{2\beta^2 m} \left\{ (1 + i\beta m \omega \hat{x}) \exp(-i\hbar \beta \hat{a}_x + (i \rightarrow -i)] \right\}.
\]

(3.11)

We now require that \( K_\beta \) act on the subspace of polynomials in \( \mathcal{H}_\beta \). (Equivalently, \( \hat{H}(\beta) \) acts on the subspace \( \Psi_0^{(\beta)}(x) \) [polynomials] in \( L^2(R, dx) \).) This requirement makes sense, since it is clear that \( K_\beta \) leaves the subspace of polynomials invariant.

More is true, however: \( K_\beta \) has a triangular action on polynomials. To be specific, one has

\[
K_\beta x^n = \frac{1}{\beta^2 m} (1 + n\hbar \beta^2 m \omega) x^n + \text{(lower order)}, \quad n \in \mathbb{N},
\]

(3.12)

as is readily verified. Next, let \( P(x), Q(x) \) be two polynomials, and consider the inner product

\[
\beta^2 m(P, K_\beta Q)_{\mathcal{H}_\beta} = \int_{-\infty}^{\infty} dx \Gamma \left( \frac{1}{\hbar \beta^2 m \omega} + \frac{i x}{\hbar \beta} \right) \Gamma \left( \frac{1}{\hbar \beta^2 m \omega} - \frac{i x}{\hbar \beta} \right) P(x)
\]

\[
\times [(1 + i\beta m \omega) Q(x - i\hbar \beta) + (i \rightarrow -i)].
\]

(3.13)

When we now shift the contour in the first term up by \( i\hbar \beta \), no singularity is crossed for \( 0 < \hbar \beta^2 m \omega < 1 \), while for \( \hbar \beta^2 m \omega \geq 1 \) the pole \( z = 0 \) of \( \Gamma(z) \) is encountered in the first \( \Gamma \)-factor. But since it is matched by the zero of \( (1 + i\beta m \omega) \), no residue is picked up. Likewise, we may shift the contour in the second term down by \( i\hbar \beta \). When we now use (3.8), we deduce

\[
(P, K_\beta Q)_{\mathcal{H}_\beta} = (K_\beta P, Q)_{\mathcal{H}_\beta}.
\]

(3.14)

The upshot is that \( K_\beta \) is triangular and symmetric on the subspace of polynomials. But then it readily follows (using (3.12)) that there are polynomials \( P_n^{(\beta)}(x) \) of degree \( n, n \in \mathbb{N} \), such that

\[
K_\beta P_n^{(\beta)} = \left( \frac{1}{\beta^2 m} + n \hbar \omega \right) P_n^{(\beta)}, \quad n \in \mathbb{N}.
\]

(3.15)
Since the eigenvalues are positive and distinct, we conclude from (3.14) that the polynomials \( \{P_n^{(\beta)}(x)\}_{n=0}^\infty \) are orthogonal in \( \mathcal{H}_\beta \). We normalize the polynomial \( P_n^{(\beta)}(x) \) by requiring that the coefficient of \( x^n \) be 1. (In particular, \( P_0^{(\beta)}(x) = 1 \).) Then it follows that we have arrived at the monic orthogonal polynomials with respect to the measure \( [\Psi_0^{(\beta)}(x)]^2 \, dx \) on the line.

The polynomials actually form a dense subspace of \( \mathcal{H}_\beta \). Accepting this assertion for the moment, we return to \( \hat{H}(\beta) \) (3.4), and conclude that our domain requirement has led to an orthogonal basis

\[
\Psi_n^{(\beta)}(x) \equiv \Psi_0^{(\beta)}(x) P_n^{(\beta)}(x), \quad n \in \mathbb{N},
\]

of eigenfunctions in \( L^2(R, \, dx) \), which satisfy

\[
\hat{H}(\beta)\Psi_n^{(\beta)} = E_n^{(\beta)}\Psi_n^{(\beta)},
\]

with

\[
E_n^{(\beta)} = \left( \frac{1}{\beta^2 m} + n\hbar \omega \right).
\]

Therefore, the operator \( \hat{H}(\beta) \) defined on the dense subspace \( \Psi_0^{(\beta)}(x) \) [polynomials] via (3.17) is essentially self-adjoint. We denote its self-adjoint closure again by \( \hat{H}(\beta) \). (Though the domain of the latter self-adjoint operator is now uniquely determined, there is no simple way to characterize it in terms of analyticity properties.)

It remains to prove our assertion that the polynomials are dense in \( \mathcal{H}_\beta \). This is not immediate, since the Stone–Weierstrass theorem only holds on compact intervals. But it is not hard to derive denseness from the exponential decay of the measure; cf. (3.9). Indeed, assuming \( \Phi(x) \in \mathcal{H}_\beta \) is orthogonal to \( x^n \) for all \( n \in \mathbb{N} \), we can prove \( \Phi = 0 \), as follows.

By virtue of the asymptotics (3.9), one has \( \exp(ixz) \in \mathcal{H}_\beta \) for \( |\text{Im} \, z| < \pi/2\beta \). Then the function \( z \mapsto (\Phi, e^{ixz})_{\mathcal{H}_0} \) is analytic for \( |\text{Im} \, z| < \pi/2\beta \). Since all of its derivatives at \( z = 0 \) vanish due to our assumption, it is identically zero. Thus the Fourier transform of \( \Phi(x)\Psi_0^{(\beta)}(x) \) vanishes. Clearly, this entails \( \Phi = 0 \), proving our assertion.

We finish this section with some comments bearing on the uniqueness of our quantization procedure. To begin with, we should admit that there exist far more obvious ways to turn the classical Hamiltonian \( H_c(x, p) \) (2.8) into a formally self-adjoint operator. Taking the Hamilton function \( |f(x)|^2 \exp(p) \) as a simple paradigm, one can in particular choose the following formally self-adjoint quantum operators:

\[
\frac{1}{2} [\exp(\hat{p})|f(x)|^2 + |f(x)|^2 \exp(\hat{p})], \quad \exp(\hat{p}/2)|f(x)|^2 \exp(\hat{p}/2),
\]

\[
|f(x)| \exp(\hat{p})|f(x)|, \quad f(x) \exp(\hat{p})\bar{f}(x), \quad f(x) \exp(\hat{p})f(x).
\]

Since \( H_c(x, p) \) is basically the sum of two such Hamiltonians, one gets even more possibilities for \( H_c(x, p) \).

When one starts from other ordering choices than the one we have made above, however, it is far from clear whether one can turn the formally self-adjoint quantum Hamiltonian into a bona fide self-adjoint operator acting on a dense domain in the Hilbert space \( L^2(R, \, dx) \). The difficulties that are involved can already be illustrated by sticking to our particular ordering choice (3.4).

As a first observation, note that whenever the eigenvalue equation (3.5) is solved by a square-integrable function \( \Psi(x) \), the function \( \Psi(x)/\cosh(\pi x/\hbar \beta) \) is also square-integrable and is an
eigenfunction with eigenvalue $-E$. Second, the function

$$
\Psi^{(p)}_0(x) \equiv \left( \Gamma \left( 1 - \frac{1}{\hbar \beta^2 m \omega} + \frac{i x}{\hbar \beta} \right) \Gamma \left( 1 - \frac{1}{\hbar \beta^2 m \omega} - \frac{i x}{\hbar \beta} \right) \right)^{1/2}
$$

(3.20)

is an eigenfunction with eigenvalue $-\hbar \omega + (m \beta^2)^{-1}$, and it is in $L^2(R)$ whenever $(\hbar \beta^2 m \omega)^{-1}$ is not equal to a positive integer. (Just like $\Psi^{(p)}_0(x)$, this eigenfunction may seem “pulled out of a hat.” In the next section we will see how both arise from an algebraic formalism.) These two observations exemplify that one cannot speak about “the spectrum” of the analytic difference operator $\hat{H}(\beta)$ until one has chosen a dense subspace in $L^2(R)$ on which its action gives rise to an essentially self-adjoint operator. Indeed, for the choice we have made above we have obtained a spectrum that is positive and does not contain the number $-\hbar \omega + (m \beta^2)^{-1}$.

Consider next the analytic difference operator with the ordering choice opposite to $\hat{H}(\beta)$, namely

$$
\hat{H}^0(\beta) \equiv \frac{1}{2 \beta^2 m} \left[ (1 - i \beta m \omega \hat{x})^{1/2} \exp(-i \hbar \beta \partial_x)(1 + i \beta m \omega \hat{x})^{1/2} + (i \rightarrow -i) \right].
$$

(3.21)

It can be viewed in two distinct ways. First, one can reinterpret it as $\hat{H}(\beta)$ (3.4), analytically continued to negative $\omega$. Now (3.7) also holds for negative $\omega$, and when $(\hbar \beta^2 m \omega)^{-1}$ is not a negative integer, $\Psi^{(p)}_0$ is still in $L^2(R)$. Thus it would seem that one can proceed as before, arriving at a self-adjoint operator whose spectrum is given by (3.18) with $\omega$ negative.

This is, however, not the case. The crux is that the operator $K_\beta$ is triangular, but not symmetric on the polynomial subspace for negative $\omega$. Accordingly, the operator $\hat{H}^0(\beta)$ (with positive $\omega$) is not symmetric on the dense subspace $\Psi^{(p)}_0(x)$ of polynomials.

A second way to look at $\hat{H}^0(\beta)$ proceeds as follows. Pushing the $\hat{x}$-dependent factors through the shifts, we obtain

$$
\hat{H}^0(\beta) = \frac{1}{2 \beta^2 m} \left[ (1 + \hbar \beta^2 m \omega + i \beta m \omega \hat{x})^{1/2} \exp(-i \hbar \beta \partial_x) \right.
\times \left. (1 + \hbar \beta^2 m \omega - i \beta m \omega \hat{x})^{1/2} + (i \rightarrow -i) \right].
$$

(3.22)

Now when we indicate the $m$-dependence explicitly, we deduce that we have

$$
\hat{H}^0(\beta, m) = \hat{H}(\beta, m'),
$$

(3.23)

where

$$
m' \equiv m/(1 + \hbar \beta^2 m \omega).
$$

(3.24)

From this viewpoint, then, the different ordering in $\hat{H}^0(\beta)$ simply leads to a mass renormalization in $\hat{H}(\beta)$. In particular, one can associate to $\hat{H}^0(\beta)$ a self-adjoint operator whose spectrum is given by (3.18) with $m \rightarrow m'$.

4. ALGEBRAIC ASPECTS OF THE OSCILLATOR FAMILY

In this section we supplement the above findings by introducing and employing an algebraic framework. The main idea is to mimic the well-known lowering and raising operator formalism...
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associated to the usual harmonic oscillator Hamiltonian (1.7). Let us begin by recalling that this formalism has a classical analog. Specifically, the function

\[ a = \left( \frac{m \omega}{2} \right)^{1/2} \left( x + i \frac{p}{m \omega} \right) \] (4.1)

and its complex conjugate \( \bar{a} \) satisfy

\[ \{a, \bar{a}\} = -i \] (4.2)

and

\[ \{a, H_E\} = -i \omega a, \quad \{\bar{a}, H_E\} = i \omega \bar{a}. \] (4.3)

It is routine to verify these Poisson brackets directly. But to lead up to our 1-parameter generalization, we point out a more conceptual way of seeing why (4.3) holds true. Indeed, in (4.1) we may replace \( p \) by \( m \ddot{x} = m \{x, H_E\}. \) The point is that in the time derivative \( \dot{a} = \{a, H_E\} \) we then obtain the term \( \dddot{x} = -\omega^2 x, \) so that (4.3) is nearly immediate from the definition of \( a. \)

Turning to our 1-parameter generalization \( H_\epsilon \) of \( H_E, \) we now recall that it also gives rise to the latter (Newton equation) term. Therefore, we obtain a natural generalization of \( a \) by setting

\[ A \equiv \left( \frac{m \omega}{2} \right)^{1/2} \left( x + \frac{i}{\omega} \{x, H_\epsilon\} \right) = \left( \frac{m \omega}{2} \right)^{1/2} \left( x + \frac{i}{\beta m \omega} \sinh(\beta p)(1 + \beta^2 m^2 \omega^2 x^2)^{1/2} \right). \] (4.4)

Indeed, as the analog of (4.3) we have

\[ \{A, H_\epsilon\} = -i \omega A, \quad \{\bar{A}, H_\epsilon\} = i \omega \bar{A}. \] (4.5)

Moreover, (4.2) is generalized to

\[ \{A, \bar{A}\} = -i \beta^2 m H_\epsilon, \] (4.6)

as is easily verified.

Proceeding to the quantum level, we may and will employ a dimensionless lowering operator

\[ a \equiv \left( \frac{m \omega}{2\hbar} \right)^{1/2} \left( \hat{x} + i \frac{\hat{p}}{m \omega} \right), \quad \hat{p} = -i \hbar \partial \hat{x}. \] (4.7)

It satisfies

\[ [a, a^\dagger] = 1, \] (4.8)

where the raising operator \( a^\dagger \) is the adjoint of \( a. \) Then we obtain

\[ \hat{H}_E = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right) = \hbar \omega \left( aa^\dagger - \frac{1}{2} \right) \] (4.9)
and the ladder relations

\[ [a, \hat{H}_E] = \hbar \omega a, \quad [a^\dagger, \hat{H}_E] = -\hbar \omega a^\dagger. \] (4.10)

Just as at the classical level, these commutators can be viewed as a consequence of writing \( \hat{p} \) as \( m \dot{x} \), and then using the operator Newton equation \([\hat{x}, \hat{H}_E], \hat{H}_E] = \hbar^2 \omega^2 \hat{x} \). Now it is straightforward to check that we also have

\[ [[\hat{x}, \hat{H}(\beta)], \hat{H}(\beta)] = \hbar^2 \omega^2 \hat{x}. \] (4.11)

(Thus, the quantum Hamiltonians \( \hat{H}_E \) (1.7) and \( \hat{H}(\beta) \) (3.4) are once more Newton equivalent.) Therefore, we should set

\[
A \equiv \left( \frac{m \omega}{2 \hbar} \right)^{1/2} \left( \hat{x} + \frac{1}{\hbar \omega} [\hat{x}, \hat{H}(\beta)] \right)
= \left( \frac{m \omega}{2 \hbar} \right)^{1/2} \left( \hat{x} + \frac{i}{2 \beta \hbar \omega} (1 + i \beta m \omega \hat{x})^{1/2} \exp(-i \hbar \beta \partial_x)(1 - i \beta m \omega \hat{x})^{1/2} - (i \rightarrow -i) \right). \] (4.12)

entailing the lowering/raising commutators

\[ [A, \hat{H}(\beta)] = \hbar \omega A, \quad [A^\dagger, \hat{H}(\beta)] = -\hbar \omega A^\dagger. \] (4.13)

Moreover, one readily checks

\[ [A, A^\dagger] = \beta^2 m \hat{H}(\beta). \] (4.14)

At this point we should stress that the above commutators are all understood in a formal sense—we are not concerned with domain issues here. Rather, our goal is to employ the analytic difference operators \( A \) and \( A^\dagger \) (and some related ones we introduce in a moment) to derive more information on eigenfunctions of \( \hat{H}(\beta) \), viewed as an analytic difference operator. In the process, we will in particular obtain salient features of the orthogonal basis vectors corresponding to the self-adjoint operator on \( L^2(\mathbb{R}) \) we have associated to \( \hat{H}(\beta) \).

To facilitate this program, it is expedient to switch to dimensionless quantities. Specifically, we employ from now on a dimensionless parameter \( \lambda \) and variable \( y \):

\[ \lambda \equiv \beta (\hbar m \omega)^{1/2}, \quad y \equiv \left( \frac{m \omega}{\hbar} \right)^{1/2} x. \] (4.15)

Then we get in particular

\[ a \rightarrow 2^{-1/2} (\dot{y} + \partial_y), \quad \hat{H}_E \rightarrow \hbar \omega (-\partial_y^2 + \dot{y}^2)/2; \] (4.16)

cf. (4.7) and (1.7).

Proceeding with our 1-parameter generalizations, we find it convenient to introduce the shift

\[ S \equiv \exp(-i \lambda \partial_y) \] (4.17)
and the two analytic difference operators
\[ T_{\pm} \equiv (1 \pm i\lambda \hat{y})^{1/2} S^{\pm 1} (1 \mp i\lambda \hat{y})^{1/2}. \] (4.18)
Together with \( i\hat{y} \), the operators \( T_{\pm} \) satisfy an \( sl(2, R) \)-type Lie algebra,
\[ [T_{\pm}, i\hat{y}] = \pm \lambda T_{\pm}, \quad [T_-, T_+] = 2\lambda^3 i \hat{y}. \] (4.19)
To avoid confusion, we also switch to lower case symbols,
\[ A \rightarrow a(\lambda), \quad A^\dagger \rightarrow a^\dagger(\lambda), \quad \hat{H}(\beta)/\hbar \omega \rightarrow h(\lambda). \] (4.20)
Thus, from now on we work with
\[ a(\lambda) = 2^{-1/2}\left( \hat{y} + \frac{i}{2\lambda}(T_+ - T_-) \right), \quad a^\dagger(\lambda) = 2^{-1/2}\left( \hat{y} - \frac{i}{2\lambda}(T_+ - T_-) \right), \] \[ h(\lambda) = (2\lambda^3)^{-1}(T_+ + T_-). \] (4.21)
From (4.13) and (4.14) we then obtain the Lie algebra
\[ [a(\lambda), h(\lambda)] = a(\lambda), \quad [a^\dagger(\lambda), h(\lambda)] = -a^\dagger(\lambda), \quad [a(\lambda), a^\dagger(\lambda)] = \lambda^2 h(\lambda), \] (4.22)
which is again of \( sl(2, R) \)-type.
In order to clarify the relation of \( a^{(i)}(\lambda) \) and \( h(\lambda) \) to the operators \( a^{(i)} \) and \( \hat{H}_E/\hbar \omega \), we observe that one has
\[ T_{\pm} = 1 \mp i\lambda \partial_y + \frac{\lambda^2}{2} (-\partial_y^2 + \hat{y}^2 - 1) + O(\lambda^3), \quad \lambda \rightarrow 0. \] (4.23)
Thus we obtain
\[ \lim_{\lambda \rightarrow 0} a(\lambda) = 2^{-1/2}(\hat{y} + \partial_y) = a, \quad \lim_{\lambda \rightarrow 0} a^\dagger(\lambda) = 2^{-1/2}(\hat{y} - \partial_y) = a^\dagger, \] \[ \lim_{\lambda \rightarrow 0}(h(\lambda) - \lambda^{-2}) = (-\partial_y^2 + \hat{y}^2 - 1)/2 = (\hbar \omega)^{-1} \hat{H}_E - 1/2. \] (4.24)
It is to be noted that the extra constant \(-1/2\) (which shifts the “vacuum energy” to 0) arises from the action of the \( O(\lambda) \)-terms in the expansion of \( S^{\pm 1} \) on the \( O(\lambda) \)-terms in the expansion of \((1 \mp i\lambda \hat{y})^{1/2}\); cf. (4.18).
We continue by noting the identities
\[ a^\dagger(\lambda)a(\lambda) = G(h(\lambda), \lambda^2), \quad a(\lambda)a^\dagger(\lambda) = G(-h(\lambda), \lambda^2), \] (4.25)
where \( G(z, t) \) is the function
\[ G(z, t) \equiv \frac{t}{2}\left( z - \frac{1}{t} \right)\left( z + \frac{1}{t} \right). \] (4.26)
This entails that when we find a solution \( \psi \) to \( a(\lambda)\psi = 0 \), then we may expect that \( \psi \) is an \( h(\lambda) \)-eigenfunction with eigenvalue \( \lambda^{-2} \) or \( 1 - \lambda^{-2} \). But this annihilation equation is still a second-order analytic difference equation, just as is the Schrödinger equation \( h(\lambda)\psi = \mu \psi \).
The simplest solution to the first order analytic difference equation \( b(\lambda) \equiv 2^{-1/2}(S^{1/2}(1 - i\lambda \hat{y})^{1/2} - S^{-1/2}(1 + i\lambda \hat{y})^{1/2}) \). (4.27)

Then one easily verifies the identity

\[
 b^\dagger(\lambda)b(\lambda) = \lambda^2 h(\lambda) - 1. \tag{4.28}
\]

The simplest solution to the first order analytic difference equation \( b(\lambda)\psi = 0 \) is given by

\[
\psi_0^{(\lambda)}(y) = (\Gamma(\lambda^{-2} + iy)\Gamma(\lambda^{-2} - iy))^{1/2}. \tag{4.29}
\]

From (4.28) we then obtain \( h(\lambda)\psi_0^{(\lambda)} = \lambda^{-2}\psi_0^{(\lambda)} \). Thus, we have recovered our ground state \( \Psi_0^{(y)}(x) \) (3.6). We also point out that one has \( a(\lambda)\psi_0^{(\lambda)} = 0 \), in accordance with (4.25) and (4.26).

The ladder relations (4.13) now entail

\[
 h(\lambda)a^\dagger(\lambda)^\nu \psi_0^{(\lambda)} = (\lambda^{-2} + n)a^\dagger(\lambda)^n \psi_0^{(\lambda)}, \quad n \in N. \tag{4.30}
\]

The eigenfunctions thus obtained are essentially our previous ones \( \Psi_n^{(y)}(x) \). (We will establish the precise relation shortly.)

The results just derived are parallel to well-known results for the usual harmonic oscillator. Even so, the equation \( a(\lambda)\psi = 0 \) is second order, so that we had occasion to introduce the auxiliary factorization in terms of \( b(\lambda) \), whose annihilation equation \( b(\lambda)\psi = 0 \) is first order. Another related feature is that other auxiliary factorizations exist, yielding different eigenfunctions.

For example, introducing

\[
 c(\lambda) \equiv 2^{-1/2}([1 + i\lambda(\hat{y} + i\lambda/2)]^{1/2}S^{1/2} - [1 - i\lambda(\hat{y} - i\lambda/2)]^{1/2}S^{-1/2}), \tag{4.31}
\]

we calculate

\[
 c^\dagger(\lambda)c(\lambda) = \lambda^2(h(\lambda) + 1 - \lambda^{-2}). \tag{4.32}
\]

The simplest solution to the first order equation \( c(\lambda)\psi = 0 \) is now

\[
 \tilde{\psi}_0^{(\lambda)}(y) = (\Gamma(1 - \lambda^{-2} + iy)\Gamma(1 - \lambda^{-2} - iy))^{1/2}. \tag{4.33}
\]

Then (4.32) entails \( h(\lambda)\tilde{\psi}_0^{(\lambda)} = (-1 + \lambda^{-2})\tilde{\psi}_0^{(\lambda)} \). This eigenfunction amounts to \( \tilde{\psi}_0^{(y)}(x) \) (3.20). It is straightforward to verify that it also satisfies \( a^\dagger(\lambda)\tilde{\psi}_0^{(\lambda)} = 0 \), in accordance with (4.25) and (4.26). (Recall that for the standard harmonic oscillator the eigenfunction \( \exp(y^2/2) \) satisfying \( a^\dagger \psi = 0 \) is not square integrable.)

Next, we calculate the similarity transform

\[
 R(\lambda) \equiv 2^{-1/2}\psi_0^{(\lambda)}(\hat{y})^{-1}a(\lambda)\psi_0^{(\lambda)}(\hat{y}) = \frac{1}{2} \left( \hat{y} \left[ 1 + \frac{1}{2}(S + S^{-1}) \right] - \frac{i}{2\lambda}(S - S^{-1}) \right). \tag{4.34}
\]
From this it follows that the functions

\[ p_n^{(2)}(y) = 2^{-n/2} \psi_0^{(2)}(y)^{-1} \left( a_1^{(2)}(\lambda) \psi_0^{(2)}(y) \right)(y) = (R(\lambda)^n \cdot 1)(y) \tag{4.35} \]

are monic polynomials of degree \( n \). (Indeed, (4.34) and (4.35) provide a Rodrigues-type formula for the polynomials.) Therefore, the functions

\[ \psi_n^{(2)} \equiv \left( 2^{-1/2} a_1^{(2)}(\lambda) \right)^n \psi_0^{(2)} = \psi_0^{(2)} \cdot p_n^{(2)} , \quad n \in \mathbb{N}, \tag{4.36} \]

amount to the eigenfunctions \( \psi_n^{(2)}(x) \) from Section 3.

The 3-term recurrence of our monic polynomials can be readily derived from the above algebraic relations. Specifically, from (4.21) we have

\[ 2^{-1/2}(a_1^{(2)}(\lambda) + a(\lambda)) \psi_n = \hat{\gamma} \psi_n^{(2)}, \tag{4.37} \]

which entails

\[ \psi_{n+1}^{(2)} + \frac{1}{2} a(\lambda)a_1^{(2)}(\lambda) \psi_{n-1}^{(2)} = \hat{\gamma} \psi_n^{(2)}. \tag{4.38} \]

When we now use (4.25), (4.26), and (4.30), and then multiply by \( 1/\psi_0^{(2)} \), we obtain

\[ p_{n+1}^{(2)}(y) + \frac{\lambda^2}{4} \left( n + \frac{2}{\lambda^2} - 1 \right) p_{n-1}^{(2)}(y) = y p_n^{(2)}(y). \tag{4.39} \]

Next, we tie in the norms of the excited states with the ground state norm \( (\psi_0^{(2)} \cdot \psi_0^{(2)})^{1/2} \) via the recurrence relation (4.39). To this end we multiply the latter by \( p_{n+1}^{(2)}(y) \psi_0^{(2)}(y)^2 \) and integrate over \( y \) to obtain

\[ (\psi_{n+1}^{(2)}, \psi_n^{(2)}) = (\psi_0^{(2)}, \psi_n^{(2)}). \tag{4.40} \]

Using then (4.39) with \( n \to n + 1 \), we get

\[ (\psi_{n+1}^{(2)}, \psi_{n+1}^{(2)}) = \frac{\lambda^2}{4} \left( n + 1 \right) \left( n + \frac{2}{\lambda^2} \right) (\psi_n^{(2)}, \psi_n^{(2)}). \tag{4.41} \]

Hence we deduce

\[ (\psi_n^{(2)}, \psi_n^{(2)}) = n! \left( \frac{\lambda}{2} \right)^{2n} \frac{\Gamma(n + 2\lambda^{-2})}{\Gamma(2\lambda^{-2})} (\psi_0^{(2)}, \psi_0^{(2)}). \tag{4.42} \]

From the above reasoning we have recovered in a self-contained way a substantial part of the known lore concerning a special type of Meixner–Pollaczek polynomials [7, 8], namely the polynomials denoted by \( P_n^{(1/\alpha)}(y; \lambda; \pi/2) \) in the survey Ref. [9]. Indeed, from Eq. (1.7.2) in the latter reference we have

\[ \int_{-\infty}^{\infty} dx \, \Gamma(\alpha + ix) \Gamma(\alpha - ix) P_m^{(\alpha)}(x; \pi/2) P_n^{(\alpha)}(x; \pi/2) = \frac{1}{2^{1-2\nu}} \pi \Gamma(n + 2\alpha)/n! \cdot \delta_{mn}, \quad \alpha > 0. \tag{4.43} \]
Since \( P_0^{(\alpha)}(x; \pi/2) = 1 \), this entails
\[
\left( \psi_0^{(\lambda)}, \psi_0^{(\lambda)} \right) = 2^{1 - 2\lambda^2} \pi \lambda \Gamma(2/\lambda^2). \tag{4.44}
\]
Also, since the coefficient of \( x^n \) in \( P_n^{(\alpha)}(x; \pi/2) \) is positive \[9\], we can compare (4.42) and (4.43) to infer
\[
P_n^{(\lambda)}(y) = n!(\lambda/2)^n P_n^{(1/\lambda)}(y/\lambda; \pi/2). \tag{4.45}
\]
Finally, we point out that the normalized ground state,
\[
\hat{\psi}_0^{(\lambda)} = \left( \psi_0^{(\lambda)}, \psi_0^{(\lambda)} \right)^{-1/2} \psi_0^{(\lambda)}, \tag{4.46}
\]
converges to the normalized ground state of the standard harmonic oscillator as \( \lambda \to 0 \):
\[
\lim_{\lambda \to 0} \hat{\psi}_0^{(\lambda)}(y) = \pi^{-1/4} \exp(-y^2/2). \tag{4.47}
\]
(Indeed, this follows from (4.29) and (4.44) by using Stirling’s formula.) Then the same result for the normalized excited states readily follows via the Rodrigues formula (4.35).

5. PREVIOUS WORK ON GENERALIZED OSCILLATORS

Apart from the idea of Newton equivalence, which has guided us to our one-parameter family of Hamiltonians, there are of course an infinity of ways to introduce generalizations of oscillator Hamiltonians involving one or several parameters. Depending on aims and perspectives, there have been a great many proposals for generalizations in the literature, some of which we have probably not come across. In the following we briefly comment on similarities and differences with our oscillators of those models we are aware of.

A first class of generalizations originates in a purely algebraic setting from Wigner’s observation \[10\] that the quantum commutation relations are not uniquely determined by their compatibility with the equation of motion. This line of reasoning leads to an approach to quantum oscillators in which there is no need to deal with the \( \hat{p} \) operator.

Indeed, the requirement that \( \hat{H} \) be the generator of time translation, together with the Newton equation of motion \( d^2\hat{x}/dt^2 + \omega^2 \hat{x} = 0 \), are sufficient to set up the algebraic relations for \( \hat{x}, \hat{v} = \frac{d\hat{x}}{dt} \) and \( \hat{H} \),
\[
\hat{v} = \frac{i}{\hbar} [\hat{H}, \hat{x}], \quad -\omega^2 \hat{x} = \frac{i}{\hbar} [\hat{H}, \hat{v}], \quad [\hat{x}, \hat{v}] = \frac{i\hbar}{m} F(\hat{H}), \tag{5.1}
\]
where the last commutator is a consequence of the Jacobi identity, and \( F(z) \) is an arbitrary function. This model is better described in terms of the lowering operator \( a \),
\[
a = (m\omega/2\hbar)^{1/2} (\hat{x} + i \hat{v}/\omega), \tag{5.2}
\]
since the commutation relation \([a, \hat{H}] = i\omega a\), and its adjoint \([a^\dagger, \hat{H}] = -i\hbar a^\dagger\), entail
\[
a^\dagger a = g(\hat{H}), \quad aa^\dagger = g(\hat{H} + i\hbar \omega), \tag{5.3}
\]
where, of course, \( g(z + i\hbar \omega) - g(z) = F(z) \).
These algebraic structures, for several choices of $g(z)$, have been discussed as generalized oscillators in Refs. [11, 12] and as deformed harmonic oscillator algebras in Ref. [13]. We would like to point out that our one-parameter family of Hamilton operators (3.4), together with the corresponding raising and lowering operators (cf. (4.12)), yield a realization of this algebraic model that has not been considered previously. (Indeed, here one has $g(z) = (z - 4mc^2)(z + 4mc^2 - h\omega)/(8mc^2h\omega)$ and $F(z) = z/4mc^2$.) Moreover, we obtain this algebra while maintaining the standard commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

Further realizations of this algebraic model are the so-called $q$-oscillators, with $q$ denoting a real parameter. These oscillators can be related to quantum groups such as $SU_q(2)$. Some early references include [14–16].

A closely related route to $q$-deformed oscillators consists in changing the basic commutation relation of $\hat{x}$ and $\hat{p}$, yielding for example the deformed Heisenberg algebra $q \hat{x} \hat{p} - \hat{p} \hat{x} = i\hbar$ [17–19], or introducing other Lie products differing from the standard commutator [20]. Once again, these harmonic oscillators differ from our family.

Next, we turn to a considerable number of papers concerning generalized oscillators that are referred to as “relativistic” by the authors involved. In all of these cases, the pertinent notion of “relativistic” differs substantially from the one explained at the end of Section 2. Two distinct types of “relativistic harmonic oscillator” are considered in Ref. [21] and in the related papers Refs. [22–24], from which further work along similar lines can be traced. The two associated Hamiltonians and their eigenfunctions are quite different from ours.

In a third “relativistic” setting, however, Hamiltonians arise that are intimately related to our family. Originally, these were introduced within the so-called quasipotential approach in quantum field theory, studied by Kadyshevsky and co-workers in various papers. More specifically, a Hamiltonian similar to ours arose from considering the $s$-wave channel in the 3-dimensional model involved. This Hamiltonian was later studied in a 1-dimensional setting, in particular in Refs. [25, 26], from which further pertinent literature can be traced.

To be specific, the relevant Hamiltonian reads

$$\hat{H}_{FT} \equiv Mc^2 \cosh \left( \frac{\hat{p}}{Mc} \right) + \frac{1}{2}M\omega^2 \hat{x} \exp \left( -\frac{\hat{p}}{Mc} \right) \hat{x}. \quad (5.4)$$

It is not obvious, but true that this Hamiltonian can be related to our Hamiltonian $\hat{H}(\beta, m)$ (3.4) by a unitary similarity transformation. Indeed, taking henceforth

$$\lambda \equiv \beta(\hbar m\omega)^{1/2} \in (0, 1), \quad (5.5)$$

we may set

$$U(\beta, m; x) \equiv \exp \left( -\frac{i\hat{x}}{\hbar\beta} \ln(\lambda^{-4} - \lambda^{-2}) \right) \Gamma \left( \frac{\lambda^{-2} - i\hat{x}}{\hbar\beta} \right) / \Gamma \left( \frac{\lambda^{-2} + i\hat{x}}{\hbar\beta} \right), \quad (5.6)$$

to obtain the transformed Hamiltonian

$$\hat{H}_U(\beta, m) \equiv U(\beta, m; \hat{x})^{1/2} \hat{H}(\beta, m)U(\beta, m; \hat{x})^{-1/2} \equiv \frac{(1 - \lambda^2)^{1/2}}{\beta^2 m} \cosh(i\hbar\beta \partial_x) + \frac{1}{2} \frac{m}{(1 - \lambda^2)^{1/2}} \omega^2 \hat{x} \exp(i\hbar\beta \partial_x) \hat{x}. \quad (5.7)$$

Comparing it to $\hat{H}_{FT}$, we obtain equality for

$$\beta \to 1/Mc, \quad m/(1 - \lambda^2)^{1/2} \to M, \quad (5.8)$$
which amounts to choosing

\[ \beta \equiv 1/Mc, \quad m = -\hbar \omega/2c^2 + [(\hbar \omega/2c^2)^2 + M^2]^{1/2}. \]  

(5.9)

In these formulas we have made the \( m \)-dependence explicit, since the \( \beta \) occurring in (5.9) deviates from our previous convention (3.1). But from a mathematical viewpoint \( \beta \) is a parameter that may be freely chosen. The point of our new choices is that we can reinterpret \( \hat{H}_{FT} \) as a self-adjoint operator. Its spectrum and the associated Meixner–Pollaczek polynomials are readily seen to coincide with the spectrum and the Meixner–Pollaczek polynomials specified in Ref. [26].

To conclude this paper we would like to mention a generalization in another direction. Recall that when an inverse-square potential is added to the harmonic oscillator Hamiltonian \( \hat{H} \) (1.7), one still obtains a nondegenerate equidistant pure point spectrum. Indeed, the resulting Hamiltonian may be viewed as a special case of Calogero’s celebrated many-particle model, which gives rise to a many-particle version of these spectral properties [27]. An analytic difference operator generalization of this model was solved explicitly by van Diejen [28], who shows that the spectral properties found by Calogero basically persist in his more general model. When his eigenfunction construction is suitably specialized, it overlaps with our construction of eigenstates in Section 3.

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